Mathematical Models and Methods in Applied Sciences

ON THE REPRESENTATION OF ENERGY AND MOMENTUM IN ELASTICITY

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In order to clarify common assumptions on the form of energy and momentum in elasticity, a generalized conservation format is proposed for finite elasticity, in which total energy and momentum are not specified *a priori*. Velocity, stress, and total energy are assumed to depend constitutively on deformation gradient and momentum in a manner restricted by a dissipation principle and certain mild invariance requirements. Under these assumptions, representations are obtained for energy and momentum, demonstrating that (i) the total energy splits into separate internal and kinetic contributions, and (ii) the momentum is linear in the velocity. It is further shown that, if the stress response is strongly elliptic, the classical specifications for kinetic energy and momentum are sufficient to give elasticity the standard format of a quasilinear hyperbolic system.

Keywords: finite elasticity, hyperbolic systems of conservation laws.

1. Introduction

In continuum mechanics the total energy of a body part is generally assumed to be the sum of an internal energy plus a kinetic energy, with the latter quadratic in the velocity; moreover, the momentum is assumed to be linear in the velocity. These expressions for total energy and momentum are, with few exceptions,[5,17–20,21–23] accepted without further discussion. One may ask, however, whether they could be derived from first principles. Such a derivation would perhaps allow for alternative expressions, and yet lead to a plausible mechanics; or, while only conceptually important for well-established continuum theories, it would suggest a *modus operandi* in the case of continuum theories where identification and representation of inertial quantities is by no means obvious (*e.g.*, theories of liquid crystals, deformable ferromagnets, moving phase boundaries, crack dynamics, adhesion and peeling, ...).

Recently, a derivation based upon invariance of the internal power and linearity of the inertial power has been proposed for classical Cauchy continua;[13] the format of this derivation is sufficiently robust to allow for various generalizations.[4,8–10,14,15] Here we follow an alternative course: we base our approach on a presumption of mathematical structure, namely, that the representations for energy and momentum in finite elasticity be such that the resulting evolution problem have the form of a weakly-hyperbolic system of conservation laws, compatible with a dissipation inequality and certain mild invariance requirements.

In Section 2 we first recall briefly the formulation of initial-value problems for general quasilinear hyperbolic systems of conservation laws.[2,3,16] Within this formulation, the standard format for nonlinear elasticity assigns the role of state variables to the deformation gradient $\mathbf{F} = \partial_x f$ and the velocity $\mathbf{v} = \partial_t f$ in a motion $f(x,t)$; further, the momentum is taken proportional to velocity, the stress $\mathbf{S} = \partial_{\mathbf{F}}\sigma$ is determined by the stored energy $\hat{\sigma}(\mathbf{F})$,[a] and the total energy τ is stored energy plus kinetic energy, with the latter proportional to $|\mathbf{v}|^2$.

We introduce next a generalization of this conservation format for elasticity, in which the total energy, the stress, and the *velocity* are given by general constitutive prescriptions $\tau = \hat{\tau}(\mathbf{F}, \mathbf{p})$ *etc.* in terms of a pair of kinematic state variables, the tensor $\mathbf{F}$ and the vector $\mathbf{p}$, to be interpreted as deformation gradient and momentum but initially unrelated to motion. As a criterium of constitutive admissibility for $\hat{\tau}(\mathbf{F}, \mathbf{p})$, $\hat{\mathbf{S}}(\mathbf{F}, \mathbf{p})$, and $\hat{\mathbf{v}}(\mathbf{F}, \mathbf{p})$, we postulate that the purely mechanical *dissipation inequality*

$$\dot{\overline{\hat{\tau}(\mathbf{F}, \mathbf{p})}} \;\le\; \hat{\mathbf{S}}(\mathbf{F}, \mathbf{p}) \cdot \dot{\mathbf{F}} + \hat{\mathbf{v}}(\mathbf{F}, \mathbf{p}) \cdot \dot{\mathbf{p}}$$

be identically satisfied in all admissible smooth processes consistent with the *conservation laws*

$$\dot{\mathbf{F}}(x,t) - \mathrm{Div}\,(\mathbf{v}(x,t) \otimes \mathbf{1}) = \mathbf{0}, \qquad \dot{\mathbf{p}}(x,t) - \mathrm{Div}\,\mathbf{S}(x,t) = \mathbf{0}. \tag{1.1}$$

In this formulation of evolutionary elasticity, momentum replaces velocity as a kinematic state variable. No relation of momentum to mass and velocity is postulated; in particular, there is no need to assume that the referential mass density be constant in order to apply the general conservation format to elasticity. The format we propose requires that quantities describing boundary fluxes be the subject of constitutive prescriptions; we then introduce such prescriptions for both stress ($\equiv$ boundary flux of momentum) and velocity ($\equiv$ boundary flux of deformation gradient).

The purpose of our generalized formulation of elasticity is to determine the constitutive specifications for energy, momentum, and stress that are compatible with both the dissipation inequality and the conservation laws. Section 3 provides these results. To obtain them, we require arbitrary local continuations $(\dot{\mathbf{F}}, \dot{\mathbf{p}})$ of a kinematic process through $(\mathbf{F}, \mathbf{p})$; since we do not include source terms in the conservation laws, we can only count on initial conditions. Two technical conditions,

[a] In elasticity, a purely mechanical context, stored energy replaces internal energy.

together akin to hyperbolicity but weaker, are to be satisfied by the constitutive mappings for velocity and stress. The first condition is

- *normality* of $\hat{\mathbf{v}}(\mathbf{F},\mathbf{p})$ with respect to $\mathbf{p}$ for each fixed $\mathbf{F}$.

A normal $\hat{\mathbf{v}}(\mathbf{F},\mathbf{p})$ has a local inverse $\hat{\mathbf{p}}(\mathbf{F},\mathbf{v})$ for each fixed $\mathbf{F}$; it follows that the stress mapping $\tilde{\mathbf{S}}(\mathbf{F},\mathbf{v}) = \hat{\mathbf{S}}(\mathbf{F},\hat{\mathbf{p}}(\mathbf{F},\mathbf{v}))$ is well-defined. The second condition is

- *ellipticity* of $\tilde{\mathbf{S}}(\mathbf{F},\mathbf{v})$ with respect to $\mathbf{F}$ for each fixed $\mathbf{v}$.

Under these assumptions, arbitrariness in local continuation is guaranteed, and thermodynamically admissible constitutive mappings are shown to satisfy

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \partial_{\mathbf{p}}\hat{\tau}(\mathbf{F},\mathbf{p}), \qquad \hat{\mathbf{S}}(\mathbf{F},\mathbf{p}) = \partial_{\mathbf{F}}\hat{\tau}(\mathbf{F},\mathbf{p}). \tag{1.2}$$

These relations are necessary precursors to the representations for velocity and energy, which are obtained under two additional constitutive assumptions:

- *Galilean variance* of velocity in a translational change in observer,
- *parity* of total energy with respect to momentum.

These lead to the representations:

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \mathbf{V}\mathbf{p}, \qquad \hat{\tau}(\mathbf{F},\mathbf{p}) = \hat{\kappa}(\mathbf{p}) + \hat{\sigma}(\mathbf{F}), \qquad \hat{\kappa}(\mathbf{p}) = \frac{1}{2}\,\mathbf{p}\cdot\mathbf{V}\mathbf{p}, \tag{1.3}$$

with $\mathbf{V}$ a symmetric, invertible, *time-independent* tensor. We conclude that, for elasticity to have the mathematical structure of a system of conservation laws compatible with a dissipation inequality, momentum and kinetic energy *must* have the representations

$$\mathbf{p} = \mathbf{M}\mathbf{v}, \qquad \kappa = \frac{1}{2}\mathbf{v}\cdot\mathbf{M}\mathbf{v}, \tag{1.4}$$

where the symmetric *mass-density tensor* $\mathbf{M}$ is the inverse of $\mathbf{V}$; moreover, since $\mathbf{M}$ inherits time-independence from $\mathbf{V}$, mass *must* be conserved. These representations are similar to the classical ones,[b] but slightly more general due to the tensorial nature of mass density.

Relations $(1.3)_2$ and (1.4)—the main result of this paper—are the same arrived at when the evolutionary problem for general Cauchy continua is put in the invariance format introduced in Ref. 13. Remarkably, the assumptions implying (1.4) are compatible with *hyperbolicity* of the system (1.1) of conservation laws if the mass-density tensor has the classical form $\mathbf{M} = \rho\mathbf{1}$ (with $\rho > 0$) and the stress mapping is strongly elliptic.

[b]Namely, the representations

$$\mathbf{p} = \rho\,\mathbf{v} \qquad \text{and} \qquad \kappa = \frac{1}{2}\,\rho\,\mathbf{v}\cdot\mathbf{v}\,,$$

with ρ the referential mass density. Classically, one also tacitly *assumes* the additive splitting $(1.3)_2$ of the total energy, as well as that the stress is the derivative of the stored energy, a consequence of $(1.2)_2$ and $(1.3)_2$.

2. Elasticity in Conservation Format

2.1. *General hyperbolic systems with involutions*

The general format of the initial-value problems for quasilinear hyperbolic systems of conservation laws is[2,3,16]

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G^\alpha(U(x,t)) \ = \ 0, \qquad (x,t) \in \mathbb{R}^m \times (0,\infty), \tag{2.1}$$

$$U(x,0) \ = \ U_o(x), \qquad x \in \mathbb{R}^m. \tag{2.2}$$

Here ∂_t stands for $\partial/\partial t$ and ∂_α for $\partial/\partial x_\alpha$; U, the *state vector*, takes values in a subset $\mathcal{O}$ of $\mathbb{R}^n$. The m constitutive maps G^α from $\mathcal{O}$ into $\mathbb{R}^n$ are supposed to be smooth and such as to satisfy the following *hyperbolicity condition*:

(H) for each fixed U and for each unit vector $\mathbf{w}$ in $\mathbb{R}^m$, the $(n \times n)$-matrix

$$\sum_{\alpha=1}^{m} w_\alpha \, \partial_U G^\alpha \tag{2.3}$$

has real proper numbers and n linearly independent proper vectors.

Generally, the systems of conservation laws of interest for applications have an associated *"entropy"–"entropy flux"* pair $(\eta, \mathbf{q})$, with the smooth mappings $\eta(U)$ and $\mathbf{q}(U)$ defined over $\mathcal{O}$ and taking values in $\mathbb{R}$ and $\mathbb{R}^m$, respectively. The role of this pair of mappings is to characterize the admissible solutions to (2.1)–(2.2) as those satisfying the generalized *"entropy condition"*

$$\partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha q^\alpha(U) \ \leq \ 0. \tag{2.4}$$

Often the conservation system (2.1) has additional geometrical structure embodied in a system of k *"involutions,"*[3] that is, of k linear differential equations

$$\sum_{\alpha=1}^{m} A_\alpha \, \partial_\alpha U = 0, \qquad A_\alpha = \ \text{a constant } (k \times n)\text{-matrix}, \tag{2.5}$$

to be satisfied by the solutions of (2.1)–(2.2) so long as they are satisfied by the initial data (2.2). In the presence of involutions the hyperbolicity condition generally needs to be reformulated; in particular, the number of linearly independent proper vectors may be less than n.

2.2. *Finite elasticity: the standard format*

The system of finite elasticity in the referential formulation is a standard example of a hyperbolic system with involutions (provided the mass density is constant). To

show that the general format just described applies, consider the elasticity system written as [3,16]

$$\dot{\mathbf{F}} - \nabla\mathbf{v} \;=\; \mathbf{0}, \qquad (\textit{compatibility condition}) \tag{2.6}$$

$$\dot{\mathbf{v}} - \mathrm{Div}\,\mathbf{S} \;=\; \mathbf{0}, \qquad (\textit{momentum balance}) \tag{2.7}$$

where, for convenience, the referential mass density is presumed to have unit value (whence the coincidence of momentum with velocity in (2.7)), and where $\mathbf{S}$ is interpreted as the Piola stress tensor.[c]

The state vector U is the pair $(\mathbf{F}, \mathbf{v})$ of a tensor $\mathbf{F}$ and a vector $\mathbf{v}$. To insure that these can be interpreted, respectively, as the deformation gradient and the velocity in a motion, that is, to insure that there is a motion $f(x, t)$ such that $\mathbf{F}$, $\mathbf{v}$, and f satisfy

$$\mathbf{F}(x,t) = \nabla f(x,t) \qquad \text{and} \qquad \mathbf{v}(x,t) = \partial_t f(x,t), \tag{2.8}$$

it is required that $\mathbf{F}$ obey the involution following from $(2.8)_1$, namely,

$$\big(\nabla(\mathbf{Fa})\big)\mathbf{b} = \big(\nabla(\mathbf{Fb})\big)\mathbf{a} \tag{2.9}$$

for all constant vectors $\mathbf{a}$, $\mathbf{b}$ (cf. (2.5)).[d]

As to the "entropy"–"entropy flux" pair, one chooses

$$\eta(\mathbf{F},\mathbf{v}) \;=\; \kappa(\mathbf{v}) + \sigma(\mathbf{F}), \tag{2.10}$$

$$\mathbf{q}(\mathbf{F},\mathbf{v}) \;=\; -\mathbf{S}^{\top}(\mathbf{F})\,\mathbf{v}, \tag{2.11}$$

with

$$\kappa(\mathbf{v}) = \frac{1}{2}\,\mathbf{v}\!\cdot\!\mathbf{v}, \tag{2.12}$$

the *kinetic energy*, $\sigma(\mathbf{F})$ the *stored energy*, and

$$\mathbf{S}(\mathbf{F}) = \partial_{\mathbf{F}}\sigma(\mathbf{F}). \tag{2.13}$$

Thus the "entropy" in this case is the total energy $(\kappa + \sigma)$ per unit volume, the "entropy flux" is the flux necessary to balance the total energy, and the generalized "entropy condition" (2.4) takes the simple form

$$\partial_t(\kappa + \sigma) - \mathrm{Div}\,(\mathbf{S}^{\top}\mathbf{v}) = 0. \tag{2.14}$$

Remarkably, this relation is consistent with the Second Law of thermodynamics, when the latter is formulated in the form of the Clausius-Duhem inequality.

[c] As is common in continuum physics, we have used a superposed dot in place of ∂_t and the nabla symbol in place of ∂_x.

[d] As is well-known, given the pair $(\mathbf{F}, \mathbf{v})$, the kinematical compatibility conditions (2.6) and (2.9) are sufficient for the existence of a local solution f of the system (2.8). Here and henceforth, unless otherwise specified a *vector* is meant to be an element of a *three-dimensional* inner-product vector space $\mathcal{V}$, and a *tensor* is a linear transformation of $\mathcal{V}$ into itself. Moreover, whenever needed or simply convenient, $\mathcal{V}$ will be thought of as being endowed with an orthonormal basis $\{\mathbf{c}_\alpha,\ \alpha = 1,2,3\}$. All components of vectors, tensors, *etc.* will be tacitly taken with respect to this basis and related constructs.

2.3. *Finite elasticity: a generalized format*

Consider now the following generalization of the standard conservation format for finite elasticity.

For the state vector $U = (\mathbf{F}, \mathbf{p})$, and for $\mathbf{1}$ the identity tensor, let the initial-value problem (2.1)–(2.2) take the form

$$\dot{\mathbf{F}}(x,t) - \mathrm{Div}\,\big(\mathbf{v}(x,t) \otimes \mathbf{1}\big) \;=\; \mathbf{0}, \tag{2.15}$$

$$\dot{\mathbf{p}}(x,t) - \mathrm{Div}\,\mathbf{S}(x,t) \;=\; \mathbf{0}, \tag{2.16}$$

$$\mathbf{F}(x,0) \;=\; \mathbf{F}_o(x), \qquad \mathbf{p}(x,0) \;=\; \mathbf{p}_o(x). \tag{2.17}$$

Next, as the generalized format suggests, let the conservation laws (2.15) and (2.16) for the state variables "deformation gradient" $\mathbf{F}$ and "momentum" $\mathbf{p}$ be supplemented by *constitutive prescriptions for velocity and stress* as functions of state:

$$\mathbf{v} \;=\; \hat{\mathbf{v}}(\mathbf{F}, \mathbf{p}), \tag{2.18}$$

$$\mathbf{S} \;=\; \hat{\mathbf{S}}(\mathbf{F}, \mathbf{p}). \tag{2.19}$$

Finally, let the generality of these constitutive prescriptions and of the additional prescription for the *total energy*:

$$\tau = \hat{\tau}(\mathbf{F}, \mathbf{p}), \tag{2.20}$$

be restricted by requiring consistency with the Second Law, under form of the dissipation inequality

$$\dot{\hat{\tau}}(\mathbf{F}, \mathbf{p}) \;\leq\; \hat{\mathbf{S}}(\mathbf{F}, \mathbf{p}) \cdot \dot{\mathbf{F}} + \hat{\mathbf{v}}(\mathbf{F}, \mathbf{p}) \cdot \dot{\mathbf{p}}, \tag{2.21}$$

to be identically satisfied in all admissible smooth processes.[e]

As to our present formulation of finite elasticity, some comments are in order. Firstly, the state variables for system (2.15)–(2.16) are a tensor $\mathbf{F}$ and a vector $\mathbf{p}$. Formally, neither $\mathbf{F}$ need be the gradient of a deformation (this geometric compatibility problem may be solved once a solution of the initial-value problem (2.15)–(2.17) has been found, but at this stage we do not require that $\mathbf{F}$ satisfy the involution (2.9)) nor that $\mathbf{p}$ need have any *a priori* specified relation with mass density and velocity. The compatibility condition (2.6) is generalized to the evolution equation (2.15), which we interpret as the balance of $\mathbf{F}$ by way of the boundary-flux tensor $\mathbf{v} \otimes \mathbf{1}$. Similarly, the momentum balance (2.7) is generalized to (2.16), the balance of $\mathbf{p}$ by way of the boundary-flux tensor $\mathbf{S}$. Less formally, we regard (2.16) as a force balance, with $-\dot{\mathbf{p}}$ the inertial force (the only external force considered here) and $\mathrm{Div}\,\mathbf{S}$ the internal force per unit referential volume.

Secondly, just as the momentum is not given any *a priori* representation, the other inertial manifestation—kinetic energy—is not given any *a priori* representation as well. In fact, neither the total energy is split into a kinetic part and an internal part nor the Piola stress is introduced by differentiation of the latter.

[e] In all constitutive mappings we leave tacit a possible explicit dependence on the space variable, but exclude any explicit time dependence.

Our next step is to introduce and exploit a set of reasonable hypotheses allowing to conclude that slight generalizations of the classical constitutive specifications for energy, momentum, and stress, are the only specifications compatible with both the conservation laws (2.15), (2.16) and the dissipation inequality (2.21).

3. Representations of Energy and Momentum

3.1. *Normality and ellipticity. Thermodynamical admissibility*

As is customary in constitutive theories of continuous media after a famous paper by Coleman & Noll,[1] the dissipation inequality restricts the choice of admissible representations of the constitutive mappings. These restrictions are derivable under the crucial assumption that any accessible kinematic state $(\mathbf{F}, \mathbf{p})$ can be reached by a path that has an arbitrary local continuation $(\dot{\mathbf{F}}, \dot{\mathbf{p}})$. Such indispensable arbitrariness is usually taken to be granted by the presence in the basic balance equations of source terms regarded as controls at our disposal. Since sources are here set to null, we revert to the other piece of data that in principle can be made to vary arbitrarily, that is, initial conditions.[f] Our first proposition establishes that such an indispensable arbitrariness is indeed granted under reasonable conditions on the balance equations and the constitutive mappings for velocity and stress.

The constitutive mappings $\hat{\mathbf{v}}$ and $\hat{\mathbf{S}}$ are taken to be continuously differentiable and to satisfy two conditions, one each. The first condition applies to $\hat{\mathbf{v}}$:

(N) (Normality) For each $\mathbf{F}$ fixed, the map $\mathbf{p} \mapsto \hat{\mathbf{v}}(\mathbf{F}, \mathbf{p})$ is *normal*, *i.e.*, the matrix $\mathbf{N}$, defined by

$$N_{ih}(\mathbf{F}, \mathbf{p}) = \partial_{p_h} \hat{v}_i(\mathbf{F}, \mathbf{p}),$$

is invertible for all $\mathbf{p}$.

This condition formalizes the expectation that momentum and velocity be locally in one-to-one correspondence, in the sense that the map $(\mathbf{F}, \mathbf{v}) \mapsto (\mathbf{F}, \mathbf{p} = \hat{\mathbf{p}}(\mathbf{F}, \mathbf{v}))$ is well-defined, with

$$\hat{\mathbf{v}}(\mathbf{F}, \hat{\mathbf{p}}(\mathbf{F}, \mathbf{v})) = \mathbf{v}.$$

Let now

$$\tilde{\mathbf{S}}(\mathbf{F}, \mathbf{v}) := \hat{\mathbf{S}}(\mathbf{F}, \hat{\mathbf{p}}(\mathbf{F}, \mathbf{v})). \tag{3.1}$$

The second condition, rather than directly to $\hat{\mathbf{S}}(\mathbf{F}, \mathbf{p})$, applies to $\tilde{\mathbf{S}}(\mathbf{F}, \mathbf{v})$:

(E) (Ellipticity) For each $\mathbf{v}$ fixed, the map $\mathbf{F} \mapsto \tilde{\mathbf{S}}(\mathbf{F}, \mathbf{v})$ is *elliptic* in the sense of Petrovsky, *i.e.*, the matrix $\mathbf{E}$, defined by

$$E_{ih}(\mathbf{F}, \mathbf{v}; \mathbf{a}) = \partial_{F_{hk}} \tilde{S}_{ij}(\mathbf{F}, \mathbf{v})\, a_j a_k \quad \text{for each vector } \mathbf{a},$$

is invertible for all $\mathbf{F}$.

[f] This idea is not new,[12,6,7,11] although we know neither of any previous use in the context of elasticity nor of anything more than a generic claim of local existence for the initial-value problem under examination.

Normality of $\hat{\mathbf{v}}(\mathbf{F},\cdot)$ and ellipticity of $\tilde{\mathbf{S}}(\cdot,\mathbf{v})$ do not imply hyperbolicity of the system (2.15)–(2.16). As discussed later in this section, some strengthening of these assumptions is needed to obtain hyberbolicity. For the time being, we consider the following *initial-value problem*:

For each $T>0$ and $x\in\mathbb{R}^3$ fixed, find $t\mapsto(\mathbf{F}(x,t),\mathbf{p}(x,t))$ such that

$$\dot{\mathbf{F}}(x,t) = \nabla\mathbf{v}(x,t), \tag{3.2}$$

$$\dot{\mathbf{p}}(x,t) = \mathrm{Div}\,\tilde{\mathbf{S}}(\mathbf{F}(x,t),\mathbf{v}(x,t)), \tag{3.3}$$

for $t\in(0,T)$ and, moreover,

$$\mathbf{F}(x,0) = \mathbf{A}+\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_o)(\mathbf{b}\otimes\mathbf{a}), \tag{3.4}$$

$$\mathbf{v}(x,0) = \mathbf{B}(\mathbf{x}-\mathbf{x}_o)+\mathbf{c}. \tag{3.5}$$

Here x_o is a given point of $\mathbb{R}^3$, $\mathbf{A}$ and $\mathbf{B}$ are two given matrices, and $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ three given vectors. This problem is obtained by supplementing (2.15)–(2.16) by the constitutive relation (3.1) and the initial conditions (3.4)–(3.5), which involve the state variables $(\mathbf{F},\mathbf{v})$; note that, due to (N), (3.4)–(3.5) induce a unique pair of initial conditions for the state variables $(\mathbf{F},\mathbf{p})$.

Proposition 1 (Arbitrariness of initial time-rates of the state variables) Assume that the constitutive mappings $\hat{\mathbf{v}}$ and $\tilde{\mathbf{S}}$ are, respectively, normal and elliptic. Moreover, assume that, for each fixed triplet $(\mathbf{A},\mathbf{a},\mathbf{c})$, the initial-value problem (3.2)–(3.5) has a classical solution up to time T for all choices of $\mathbf{B}$, $\mathbf{b}$, and x_o. Then the initial time-rate $\big(\dot{\mathbf{F}}(x_o,0),\dot{\mathbf{p}}(x_o,0)\big)$ at x_o of the state vector can be assigned arbitrary values.

Proof. Note firstly that the evolution equations (3.2)–(3.3) can be written as

$$\dot{F}_{ij} = \partial_j v_i, \tag{3.6}$$

$$\dot{p}_i = \partial_{F_{hk}}\tilde{S}_{ij}(\mathbf{F},\mathbf{v})\,\partial_j F_{hk}+\partial_{v_k}\tilde{S}_{ij}(\mathbf{F},\mathbf{v})\,\partial_j v_k. \tag{3.7}$$

Moreover, the initial conditions (3.4)–(3.5) yield

$$(\mathbf{F}(x_o,0),\mathbf{v}(x_o,0)) = (\mathbf{A},\mathbf{c}), \tag{3.8}$$

$$(\partial_j F_{hk}(x_o,0),\partial_j v_k(x_o,0)) = (b_h a_k a_j, B_{kj}). \tag{3.9}$$

Then, since the initial-value problem (3.2)–(3.5) has classical solutions, with the notations introduced in the statement of condition (E) we have that

$$\dot{F}_{ij}(x_o,0) = B_{ij}, \tag{3.10}$$

$$\dot{p}_i(x_o,0) = \partial_{v_k}\tilde{S}_{ij}(\mathbf{A},\mathbf{c})\,B_{kj}+E_{ih}(\mathbf{A},\mathbf{c};\mathbf{a})\,b_h. \tag{3.11}$$

When $\mathbf{B}$ and $\mathbf{b}$ are arbitrarily varied, the invertibility of $\mathbf{E}(\mathbf{A},\mathbf{c};\mathbf{a})$ guarantees that the second term on the right of the last relation and, hence, $\dot{\mathbf{p}}(x_o,0)$ take arbitrary values, not only $\dot{\mathbf{F}}(x_o,0)$. $\square$

Proposition 2 (Thermodynamically admisssible constitutive relations) Under the assumptions of Proposition 1, necessary and sufficient conditions for the dissipation inequality (2.21) to be satisfied in all admissible smooth processes of the materials described by the constitutive relations (2.18)–(2.19) are

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) \;=\; \partial_{\mathbf{p}}\hat{\tau}(\mathbf{F},\mathbf{p}), \tag{3.12}$$

$$\hat{\mathbf{S}}(\mathbf{F},\mathbf{p}) \;=\; \partial_{\mathbf{F}}\hat{\tau}(\mathbf{F},\mathbf{p}). \tag{3.13}$$

Moreover, if $\hat{\tau}$ is twice continuosly differentiable, then

$$\partial_{\mathbf{p}}\hat{\mathbf{S}} = \partial_{\mathbf{F}}\hat{\mathbf{v}}. \tag{3.14}$$

Proof. From (2.21) and by the chain rule,

$$(\partial_{\mathbf{F}}\hat{\tau} - \hat{\mathbf{S}})\cdot\dot{\mathbf{F}} + (\partial_{\mathbf{p}}\hat{\tau} - \hat{\mathbf{v}})\cdot\dot{\mathbf{p}} \;\leq\; 0. \tag{3.15}$$

Sufficiency of (3.11)–(3.12) follows by substitution. Since $(\dot{\mathbf{F}},\dot{\mathbf{p}})$ can be chosen arbitrarily, at least initially (Proposition 1), we obtain necessity. □

3.2. *Galilean variance and parity. Representation results*

It is common in mechanics to impose some type of invariance requirements on the constitutive relations. Here we consider a notion of *translational change in observer* appropriate to our present choice of the state vector. Since neither $\mathbf{F}$ nor $\mathbf{p}$ are presumed to have any relation to the motion, we cannot deduce their variance laws from the standard notion of observer change. Instead, we say that the pairs $(\mathbf{F},\mathbf{p})$ and $(\mathbf{F}^*,\mathbf{p}^*)$ are related by a translational change in observer whenever there is a vector $\mathbf{d}$, independent of time, such that

$$\begin{aligned} \mathbf{F} \mapsto \mathbf{F}^* \;&=\; \mathbf{F}, \\ \mathbf{p} \mapsto \mathbf{p}^* \;&=\; \mathbf{p} + \mathbf{d}. \end{aligned} \tag{3.16}$$

Thus, while $\mathbf{F}$ is invariant, $\mathbf{p}$ is not; $\mathbf{d}$ measures the "invariance defect" of the latter. Note that when the momentum $\mathbf{p}$ is given the classical representation $\mathbf{p} = \rho\mathbf{v}$, this notion reduces to the standard one, provided one chooses $\mathbf{d} = \rho\mathbf{e}$, with $\mathbf{e}$ a constant vector. As to the behavior in a translational observer change of the constitutive mapping delivering the velocity, we introduce the following assumption.

(G) (Galilean Variance of Velocity) In a translational change in observer,

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}+\mathbf{d}) - \hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \hat{\mathbf{c}}_{\mathbf{v}}(\mathbf{d}) \qquad \text{for each vector } \mathbf{d}, \tag{3.17}$$

where the mapping $\hat{\mathbf{c}}_{\mathbf{v}}$ specifies the invariance defect of the velocity mapping $\hat{\mathbf{v}}$.

Our next assumption specifies the parity of total energy with respect to the second state variable, the momentum $\mathbf{p}$:

(P) (Parity of Total Energy with Respect to Momentum) For each $\mathbf{F}$ fixed, the total energy mapping is even in its second argument, namely,

$$\hat{\tau}(\mathbf{F},\mathbf{p}) = \hat{\tau}(\mathbf{F},-\mathbf{p}). \tag{3.18}$$

Note that, due to (3.11), parity of total energy renders the velocity odd:

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = -\hat{\mathbf{v}}(\mathbf{F},-\mathbf{p}). \tag{3.19}$$

The next two propositions state our main representation results.

Proposition 3 (Constitutive representation of velocity) Let the constitutive mappings $\hat{\mathbf{v}}$ and $\hat{\tau}$ satisfy, respectively, the variance and parity requirements (G) and (P). Then $\hat{\mathbf{v}}$ has the representation

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \mathbf{V}\mathbf{p}, \tag{3.20}$$

where the symmetric and invertible tensor $\mathbf{V}$ depends at most on x.

Proof. By assumption (G),

$$\begin{aligned} \partial_{\mathbf{F}}\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}+\mathbf{d}) &= \partial_{\mathbf{F}}\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}), &(3.21)\\ \partial_{\mathbf{p}}\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}+\mathbf{d}) &= \partial_{\mathbf{p}}\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}), &(3.22) \end{aligned}$$

so that, both $\partial_{\mathbf{F}}\hat{\mathbf{v}}$ and $\partial_{\mathbf{p}}\hat{\mathbf{v}}$ do not depend on $\mathbf{p}$. Relation (3.21) then yields the preliminary representation

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \tilde{\mathbf{V}}(\mathbf{F})\mathbf{p} + \tilde{\mathbf{v}}(\mathbf{F}), \tag{3.23}$$

which, by consequence (3.18) of assumption (P), reduces to

$$\hat{\mathbf{v}}(\mathbf{F},\mathbf{p}) = \tilde{\mathbf{V}}(\mathbf{F})\mathbf{p}. \tag{3.24}$$

Consistency with (3.20) requires that $\tilde{\mathbf{V}}(\mathbf{F}) \equiv \mathbf{V}$, with $\mathbf{V}$ depending at most on x. Moreover, the normality assumption (N) guarantees that the tensor $\mathbf{V}$ is invertible. Finally, it follows from (3.11) and (3.23) that $\mathbf{V}$ is symmetric. $\square$

Proposition 4 (Constitutive representation of energy) If Propositions (1)–(3) hold, then the total energy $\hat{\tau}(\mathbf{F},\mathbf{p})$ has the representation

$$\hat{\tau}(\mathbf{F},\mathbf{p}) = \hat{\kappa}(\mathbf{p}) + \hat{\sigma}(\mathbf{F}), \qquad \hat{\kappa}(\mathbf{p}) = \frac{1}{2}\,\mathbf{p}\cdot\mathbf{V}\mathbf{p}, \tag{3.25}$$

where $\hat{\kappa}(\mathbf{p})$ is interpreted as the *kinetic energy* and $\hat{\sigma}(\mathbf{F})$ as the *stored energy.*

Proof. By (3.12), (3.13), and (3.19), we have that

$$\partial^{(2)}_{\mathbf{F}\mathbf{p}}\hat{\tau}(\mathbf{F},\mathbf{p}) = \mathbf{0}, \tag{3.26}$$

which yields $(3.25)_1$. The representation $(3.25)_2$ then follows from (3.11) and again (3.19). $\square$

Because of the presence of the tensor $\mathbf{V}$, the expressions for $\hat{\mathbf{v}}$ and $\hat{\kappa}$ are slightly more general than the usual ones. As expected, the stored energy determines the stress, since (3.13) and $(3.25)_1$ immediately yield

$$\hat{\mathbf{S}}(\mathbf{F}) = \partial_{\mathbf{F}}\hat{\sigma}(\mathbf{F}). \tag{3.27}$$

We note that, since (the velocity mapping is independent of $\mathbf{F}$ and) the stress mapping is independent of $\mathbf{p}$, the ellipticity condition (E) can be phrased in terms of $\hat{\mathbf{S}}(\mathbf{F}, \mathbf{p})$.

3.3. *Constitutive representations and hyperbolicity*

We have proposed a set of reasoned assumptions under which the general constitutive relations (2.18), (2.19), and (2.20) take, respectively, the forms (3.20), (3.27), and (3.25). The latter group of relations implies hyperbolicity of the elasticity system, provided further assumptions are introduced. Such hyperbolicity—which, as is well-known, cannot be strict—is described precisely in our last proposition.

As a premise, we note that, for

$$\mathbb{S} = \partial_{\mathbf{F}}\hat{\mathbf{S}} = \partial^{(2)}_{\mathbf{FF}}\hat{\sigma}\,, \tag{3.28}$$

the ellipticity matrix can be written as a linear transformation of vectors:

$$\mathbf{E}(\mathbf{w})\mathbf{u} = (\mathbb{S}[\mathbf{u}\otimes\mathbf{w}])\mathbf{w}, \qquad \mathbf{w}, \mathbf{u} \in \mathcal{V}, \quad |\mathbf{w}| = 1. \tag{3.29}$$

Due to (3.28), *strong ellipticity* of the stress mapping $\hat{\mathbf{S}}$ (that is, positivity of $\mathbf{E}(\mathbf{w})$ for each unit vector $\mathbf{w}$) is equivalent to *strict* (*rank 1*)-*convexity* of the stored-energy mapping $\hat{\sigma}$.

Proposition 5 (Hyperbolicity of Elasticity in Conservation Format) Let $\mathbf{V} = \rho^{-1}\mathbf{1}$, where $\rho > 0$ is interpreted as the referential *mass density*. Moreover, let the matrix $\mathbf{E}(\mathbf{w})$ be positive for each unit vector $\mathbf{w}$. Then, for each $\mathbf{w}$, the (12×12)-matrix that now corresponds to the general hyperbolicity matrix (2.3) has zero as proper value of geometric multiplicity six, and has, moreover, exactly three independent proper vectors corresponding to non-null proper numbers and giving the possible directions of wave propagation.[g]

Proof. Under the assumptions, for each fixed $\mathbf{w}$, the problem we study can be given the form

$$\mathbb{M}\,(\mathbf{w})\,V = \lambda\,V, \tag{3.30}$$

[g] The occurrence of zero as a proper number with high multiplicity is due to the involutions (2.9), which render the elasticity system not strictly hyperbolic (cf. Dafermos[3]).

where the entries of the (4×4)-blockmatrix $\mathbb{M}$ that satisfies (2.3) are the following tensors:

$$\left.\begin{aligned} \mathbb{M}_{\alpha\beta} &= \mathbb{M}_{44} = \mathbf{0}, && (\alpha,\ \beta = 1,2,3), \\ \mathbb{M}_{\alpha 4} &= -\rho^{-1}\mathbf{w}\otimes\mathbf{c}_\alpha, && \\ \mathbb{M}_{4\alpha}\mathbf{u} &= -(\mathbb{S}[\mathbf{u}\otimes\mathbf{c}_\alpha])\mathbf{w}, && \mathbf{u}\in\mathcal{V}, \end{aligned}\right\} \tag{3.31}$$

and where, given a tensor $\mathbf{Z}$ and a vector $\mathbf{z}$, the entries of the 4-blockvector V are the vectors

$$\mathbf{v}_\alpha = \mathbf{Z}\mathbf{c}_\alpha \ \ (\alpha = 1,2,3), \qquad \mathbf{v}_4 = \mathbf{z}. \tag{3.32}$$

With (3.31) the equation (3.30) yields the system

$$\mathbb{M}_{\alpha 4}\mathbf{v}_4 = \lambda\mathbf{v}_\alpha, \tag{3.33}$$

$$\sum_{\alpha=1}^{3}\mathbb{M}_{4\alpha}\mathbf{v}_\alpha = \lambda\mathbf{v}_4, \tag{3.34}$$

or rather, more explicitly,

$$-\rho^{-1}\mathbf{z}\otimes\mathbf{w} = \lambda\mathbf{Z}, \tag{3.35}$$

$$-(\mathbb{S}[\mathbf{Z}])\mathbf{w} = \lambda\mathbf{z}. \tag{3.36}$$

For $\lambda = 0$, (3.35) implies that $\mathbf{z} = \mathbf{0}$; moreover, choosing $\mathbf{c}_3 = \mathbf{w}$, we can represent $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{v}_1\otimes\mathbf{c}_1 + \mathbf{v}_2\otimes\mathbf{c}_2 + \mathbf{v}_3\otimes\mathbf{w}; \tag{3.37}$$

consequently, (3.36) becomes

$$(\mathbb{S}[\mathbf{v}_3\otimes\mathbf{w}])\mathbf{w} = \mathbf{E}(\mathbf{w})\mathbf{v}_3 = -(\mathbb{S}[\mathbf{v}_1\otimes\mathbf{c}_1 + \mathbf{v}_2\otimes\mathbf{c}_2])\mathbf{w}, \tag{3.38}$$

whence, in view of the assumed positivity of $\mathbf{E}(\mathbf{w})$, we conclude that $\mathbf{Z}$ belongs to a six-dimensional subspace of the space of all tensors. Moreover, assuming provisionally that $\lambda \neq 0$, one finds by inserting (3.35) into (3.36) that

$$\mathbf{E}(\mathbf{w})\mathbf{z} = \mu\,\mathbf{z}, \qquad \mu = \rho\,\lambda^2. \tag{3.39}$$

Hence, the positivity of $\mathbf{E}(\mathbf{w})$ implies that there are *positive* proper numbers μ and exactly three independent proper vectors. $\square$

We remark that the involution condition (2.9)—which we do not use—implies that the proper space associated to the proper number $\lambda = 0$ is six-dimensional.[3]

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